CS103X: Discrete Structures
Homework Assignment 6

Due March 7, 2008

Exercise 1 (10 points). How many simple directed (unweighted) graphs on the set of vertices $\{v_1, v_2, \ldots, v_n\}$ are there that have at most one edge between any pair of vertices? (That is, for two vertices a, b, only at most one of the edges (a, b) and (b, a) is in the graph.) For this question vertices are distinct and isomorphic graphs are not the same. Substantiate your answer.

Solution Between any two vertices v_a and v_b, there are three possibilities: v_a and v_b are not connected, $v_a \to v_b$, $v_b \to v_a$. For a graph with n vertices, there are $\binom{n}{2}$ ways to pair up vertices and this leads to a total of $3^{\binom{n}{2}}$ different directed graphs.

Exercise 2 (20 points). Given a connected graph $G = (V, E)$, the *distance* $d_G(u, v)$ of two vertices u, v in G is defined as the length of a shortest path between u and v. The *diameter* $\text{diam}(G)$ of G is defined as the greatest distance among all pairs of vertices in G. (That is, $\max_{u,v \in V} d_G(u, v)$.) The *eccentricity* $\text{ecc}(v)$ of a vertex v of G is defined as $\max_{u \in V} d_G(u, v)$. Finally, the *radius* $\text{rad}(G)$ of G is defined as the minimal eccentricity of a vertex in G, namely $\min_{v \in V} \text{ecc}(v)$. Prove:

(a) $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$.

(b) For every $n \in \mathbb{N}^+$, there are connected graphs G_1 and G_2 with $\text{diam}(G_1) = \text{rad}(G_1) = n$ and $\text{diam}(G_2) = 2\text{rad}(G_2) = 2n$.

Solution

(a) As $\text{rad}(G) = \min_{v \in V} [\max_{u \in V} d_G(u, v)]$, so obviously $\text{rad}(G) \leq \text{diam}(G)$
Now suppose that $\text{diam}(G)$ goes from vertices d_1 to d_2, $d_1, d_2 \in V$. Recall that $\text{rad}(G) = min_{v \in V} \text{ecc}(v)$. Let the chosen v for minimal eccentricity be v^*.
Note that $\text{diam}(G) \leq d_G(v^*, d_1) + d_G(v^*, d_2)$. Since $\text{diam}(G)$ is the shortest path from d_1 to d_2, any other path from d_1 to d_2 is either as long or longer. Also note that $d_G(v^*, d_1) + d_G(v^*, d_2) \leq \text{rad}(G) + \text{rad}(G) = 2\text{rad}(G)$ since $\text{rad}(G)$ is the maximum distance of any other vertex from $v*$ in G. Hence,
$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$$

(b) Consider a cycle with $2n$ or $2n + 1$ vertices. This will always have $\text{diam}(G_1) = \text{rad}(G_1) = n$. For $\text{diam}(G_2) = 2\text{rad}(G_2) = 2n$, consider a line graph with $2n + 1$ vertices.

Exercise 3 (10 points). Let G be a graph in which all vertices have degree at least d. Prove that G contains a path of length d.

Solution Let the longest path have length p. Consider the last vertex in the path. It has degree at least d, therefore, they must all be in the path otherwise we can make a longer path by adding any of those. Therefore, the longest path must include at least $d + 1$ vertices, meaning the longest path must be at least length d, so a path of length d can be found by taking a subpath of the longest path.

Exercise 4 (15 points)**.** Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Prove that if all vertex degrees in a graph G are even then G has no bridge.

Solution We may assume that G is connected, for otherwise the lemma could be applied to each component separately. For contradiction, suppose that an edge $\{v_1, v_2\} = e$ is a bridge of G. The graph $G' = (V, E \setminus \{e\})$ has exactly 2 components. Let G_1 be the component containing v_1. All vertices of G_1 have an even degree except for v_1 whose degree in G_1 is odd. But this is impossible by the handshake lemma.

Exercise 5 (10 points)**.** Prove that given a connected graph $G = (V, E)$, the degrees of all vertices of G are even if and only if there is a set of edge-disjoint cycles in G that cover the edges of G. (That is, the edge set of G is the disjoint union of the edge sets of these cycles.)

Solution We would prove this by strong induction on the number of vertices. For the induction basis, consider a graph with with a single vertex and the proposition holds trivially. Assume that this holds for all graphs with up to n vertices for $n \geq 2$. Now consider a graph G with $n + 1$ vertices. Since each vertex in G is even and of degree at least 2, so G is not a tree (no vertex of degree 1). Thus, there is at least one cycle C in the graph. If G is not this cycle, let G' be the subgraph (possibly disconnected) obtained from G by deleting all the edges belonging to C. Since every vertex in a cycle is of degree 2 and every vertex in G' is also even, by the induction hypothesis G' has a set of edges that is the disjoint union of edge sets of cycles. Thus, the set of edges of G will be the disjoint union of edge sets of G' and the deleted cycle.
Conversely, consider a graph with a single vertex (set of edges is empty). Obviously, the vertex has an even degree. Assume that this holds for all graphs with up to n vertices for $n \geq 2$. Now consider a connected graph G with $n + 1$ vertices such that the set of edges in G is the disjoint union of m cycles. Consider any one of these cycles, say C. Since G is connected, there is a vertex in common between C and the rest of the graph G', obtained by omitting the edges in cycle C from the set of edges of G. Since every vertex in a cycle has degree 2, and by our induction hypothesis all vertices in G_1 have even degrees, all vertices in G will have even degrees. This concludes the proof.

Exercise 6 (10 points)**.** Given a graph G, its *line graph* $L(G)$ is defined as follows:

- Every edge of G corresponds to a unique vertex of $L(G)$.
- Any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint.

Prove that if G is regular and connected then $L(G)$ is Eulerian.

Solution If the degree of regular graph G is d, then every edge of G has $2(d - 1)$ neighbours in $L(G)$. Since this is even, $L(G)$ is Eulerian.

Exercise 7 (10 points)**.** Prove that if a graph has at most m vertices of degree at most n and all other vertices have degree at most k, with $k < n$ and $m < n$, then the graph is colorable with $m + k + 1$ colors.

Solution First consider the reduced problem of coloring the graph minus the m vertices of degree at most n and all edges involving those vertices. From the lecture notes, since all remaining vertices have degree k or less, $k + 1$ colors are enough for this reduced graph. Then if we restore the original graph and assign one color not used to each of the m vertices, the resulting graph can be colored using $m + k + 1$ colors.

Exercise 8 (15 points)**.** Let G be a simple graph with n vertices. Prove that if G does not have K_3 as an induced subgraph, G has at most $\lfloor \frac{n^2}{4} \rfloor$ edges.

Solution Let k be the maximal degree of G. Find the vertex of maximal degree v (pick any if there are more than one). This produces k edges. None of the neighbors of v are connected to each other as it would produce a 3-cycle, so they are connected to at most $n-k$ vertices. The other $n-k$ vertices (including v) are connected to at most k vertices. Thus the total number of edges is bounded by $1/2*(k(n-k)+k(n-k)) = k(n-k)$. Maximizing happens when $k = n/2$ which produces the desired bound.