CS103X: Discrete Structures
Homework Assignment 5

Due February 29, 2008

Exercise 1 (20 points). Prove:

(a) In a collection of $n + 1$ integers, at least two are congruent modulo n.
   Solution There are only n possible remainders when dividing with n. By Pigeonhole Principal, out of $n + 1$ integers at least 2 are congruent modulo n.

(b) Among any five points with integer coordinates in the plane, there are two, such that the center of the line segment that connects them also has integer coordinates.
   Solution There are four possible cases for any point p in a plane: (Even, Even), (Even, Odd), (Odd, Even), (Odd, Odd). By pigeonhole principle, at least two of them must have the same parity. Since Even + Even = Even, and Odd + Odd = Even, if there are two points p_1,p_2 with the same parity, we will have $p_1 + p_2 =$ (Even, Even), which means p_1 and p_2 have a midpoint.

(c) At a cocktail party with at least two people, there are two that know the same number of party attendees.
   Solution In a group of N people, a person may have $0, 1, 2, ..., N - 1$ friends. Assume to the contrary that all N people have different number of friends. Then for each number in the sequence $0, 1, 2, ..., N - 1$ there must be a person with exactly this number of friends. In particular, there is at least one with $N - 1$ friends. But, if so, all others have this person as a friend, implying that there is no one with no friends at all. Therefore, the only possible numbers of friends come from the shortened sequence: $1, 2, 3, ..., N - 1$. By the Pigeonhole Principle, there are at least two with the same number of friends, so that our assumption that this is not true proved wrong, thus it must be indeed true.

(d) Among five points chosen from within a square with side length 1, at least two lie within distance $\sqrt{2}/2$.
   Solution Partition the unit square into 4 squares, each with side $1/2$. Each of the 5 points lies in one of these 4 squares. (If a point lies on the boundary between squares, assign it to a square arbitrarily.) Therefore, by the Pigeonhole Principle, there exist 2 points within the same square, the maximum possible distance between them being $\sqrt{2}/2$.

Exercise 2 (10 points). A stressed-out computer science professor consumes at least one espresso every day of a particular year, drinking 500 overall. Prove that on some consecutive sequence of whole days exactly 100 espressos were consumed.

Solution Let a_i be the total number of drinks consumed up to and including the ith day, for $i = 1, \ldots, 365$. Combine these with the numbers $a_1 + 100, \ldots, a_{365} + 100$, providing 730 numbers, all positive and less than or equal to $500 + 100 = 600$. Hence two of these 730 numbers are identical. Since all ai's and, hence, $(a_i + 100)$'s are distinct, then $a_j = a_i + 100$, for some $i < j$. Thus, on days $i + 1$ to j, the person consumes exactly 100 drinks.

Exercise 3 (10 points). A betting game is played by filling out a number of betting cards. Each time you fill out a card, you pick 4 natural numbers ranging from 1 to 16. After filling out some number of cards, you hand them in and the house picks 4 of the numbers to be "losers" at random. A card is a winner if it has none of those "loser" numbers. Prove that if you fill out 6 cards, it is always possible that none of them will win.

Solution   To do this, find a four-member set that has at least one element in common with the six sets representing the cards. If any one number appears on three cards this is trivial, pick that one, and one from each other set. If this is not the case, there must be 8 numbers appearing in two sets and 8 appearing in one. Pick a number appearing in two sets, then the union of those two sets is at most 7 distinct numbers. Thus another number can picked that is in two different sets, and then pick one from each of the 2 remaining sets.

Exercise 4 (10 points). Prove that if every point on a line is painted cardinal or white, there exists three points of the same color such that one is the midpoint of the line segment formed by the other two.

Solution   Pick two points A, B of the same color. Let C be the midpoint of AB, and position D, E such that C is also the midpoint of DE and $DA = AB = BE$. If C, D, E are the same color then we're done; if not, then at least one of them is the same color as A, B and forms a trio with A, B.

Exercise 5 (10 points). Prove that at a cocktail party with ten or more people, there are either three mutual acquaintances or four mutual strangers.

Solution   Consider a person A and the 9 other people in the room. First case, suppose A has at least 4 acquaintances then if any of those acquaintances know each other, we have 3 mutual friends. If none of those acquaintances know each other, we have at least four mutual strangers, namely A's acquaintances. Consider the other case where A has more than 5 strangers (less than 4 acquaintances). Then, consider a person B in the set of A's strangers. B has at least 3 acquaintances or 3 strangers. Suppose B has at least 3 acquaintances, then none of these acquaintances know each other, otherwise we have 3 mutual acquaintances. But since A does not know any of these people, B's 3 acquaintances and A form a group of 4 mutual strangers. On the other hand, if B has at least 3 strangers, then if these strangers all know each other, we would have 3 mutual acquaitances. Thus, suppose there are at least 2 people of B's strangers that do not know each other. But then we would have these 2 people who don't know each other, B, and A as a group of mutual strangers. In all cases, we get either three mutual acquaintances or four mutual strangers, as desired.

Exercise 6 (20 points). Order the following functions by asymptotic growth rates. That is, list them as $f_1(n), f_2(n), \ldots, f_{10}(n)$, such that $f_i(n) = O(f_{i+1}(n))$ for all i. Give a brief argument to support the inequality $f_i(n) = O(f_{i+1}(n))$ for every i. (You don't have to give all the details.)

$$n^2, \log_{10} n, n \log_2 n, n^{\ln n}, (\ln n)^n, \ln(n^n), (\ln n)^{\ln n}, (\ln n)^{\ln \ln n}, (\ln \ln n)^{\ln n}, 3^{n/2}.$$

Hint: Take logarithms to bring the exponents down. Use basic properties of logarithms.

Solution
$$\log_{10} n, (\ln n)^{\ln \ln n}, \ln(n^n), n \log_2 n, n^2, (\ln \ln n)^{\ln n}, (\ln n)^{\ln n}, n^{\ln n}, 3^{n/2}, (\ln n)^n.$$

For an example, we will show $(\ln n)^{\ln \ln n} = O(\ln(n^n))$. Let $n = e^{e^k}$, then we need to show:

$$(\ln e^{e^k})^{\ln \ln e^{e^k}} = O(e^{e^k} \ln(e^{e^k}))$$

$$(e^k)^k = O(e^{e^k} e^k)$$

$$e^{2k} = O(e^{e^k + k})$$

Which is obviously true since $e^{2k} \le e^{e^k + k}\ \forall k \ge 0$.

Another example, we will show $n^2 = O((\ln n)^{\ln n})$. We need to show

$$\exists c > 0 \text{ such that } n^2 \le c(\ln n)^{\ln n}\ \forall n \ge n_0$$

First take logs on both sides

$$2\ln n \le \ln c + \ln n \ln \ln \ln n$$

Let us choose $n_0 = e^{e^2}$, then $\ln \ln n_0 = 2$. Then the above inequality holds for all $n \ge n_0$

Exercise 7 (10 points). What's wrong with the following induction proof?

> Define $f(n) = n^2$. We prove by induction that $f(n) = O(n)$. When $n = 1$, $1^2 = 1$ and the claim holds. Assume that $k^2 = O(k)$ for some $k \ge 1$. Then $(k+1)^2 = k^2 + 2k + 1$. Each of the summands on the right side is $O(k)$, and thus $(k+1)^2 = O(k)$. This completes the proof by induction.

Solution $O(n)$ is shorthand for

$$\exists c > 0 \text{ such that } f(n) \le cn \ \forall n \ge n_0$$

We are inducting over n, so we are basically setting $n_0 = 1$ (then if we can show that $f(n) \le cn$ for all $n > 1$, then we are done). The problem is in the induction step. Let us assume that $k^2 \le cn$ for some c, $k > 1$. Now consider for some $p \ge 1$, $(k+p)^2 = k^2 + 2kp + 1 \le ck + 2kp + 1$. However, if we can find some $p = p_0$ large enough such that $2kp + 1 \ge k$, then $(k+p_0)^2 = k^2 + 2kp + 1 \le ck + k = (c+1)k$, which is larger than ck. Thus, in order for $(k+p_0)$ to be $O(k)$, we need to choose a new c, namely $c+1$. Since we can find a p_0 for any k, we need to continually choose a new c as k increases. Thus, there does not exist a fixed c for all $n \ge n_0$.

Exercise 8 (10 points). Consider a relation $\propto$ on the set of functions from $\mathbb{N}^+$ to $\mathbb{R}$, such that $f \propto g$ if and only if $f = O(g)$. Is $\propto$ an equivalence relation? A partial order? A total order? Prove.

Solution $\propto$ is not an equivalence relation because it does not satisfy the symmetric property. Consider $f(n) = n$ and $g(n) = n^2$. Then $f \propto g$, but $g \propto f$ does not hold.

$\propto$ is not a partial order because it does not satisfy the antisymmetric property. Suppose $f(n) = n+1$ and $g(n) = n+2$, then $f \propto g$ and $g \propto f$ but $f \ne g$. Since $\propto$ is not a partial order, it cannot be a total order.