# CS103X: Discrete Structures
# Homework Assignment 1

Due January 25, 2008

**Exercise 1** (10 Points). Prove or give a counterexample for each of the following:

(a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(b) If $A \in B$ and $B \in C$, then $A \in C$.

**Exercise 2** (10 Points). If $a(t)$, $b(t)$, and $c(t)$ are the lengths of the three sides of a triangle $t$ in *non-decreasing order* (i.e. $a(t) \leq b(t) \leq c(t)$), we define the sets:

- $X := \{\text{Triangle } t : a(t) = b(t)\}$
- $Y := \{\text{Triangle } t : b(t) = c(t)\}$
- $T :=$ the set of all triangles

Using only set operations on these three sets, define:

(a) The set of all equilateral triangles (all sides equal)

(b) The set of all isosceles triangles (at least two sides equal)

(c) The set of all scalene triangles (no two sides equal)

**Exercise 3** (10 Points) The Fibonacci sequence is defined as follows:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 \\ a_n &= a_{n-1} + a_{n-2} \text{ for all } n \geq 3 \end{aligned}$$

Prove that

$$a_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

**Exercise 4** (10 Points) Prove by induction:

(a)

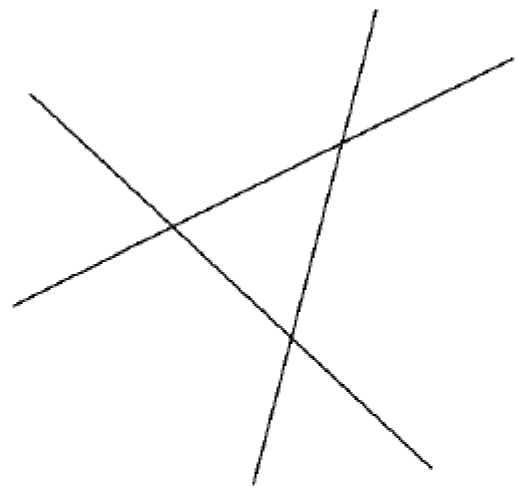
$$\sum_{i=1}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$$

(b)

$$\sum_{i=1}^{n} i^3 = \left( \sum_{i=1}^{n} i \right)^2$$

**Exercise 5** (10 Points) Consider $n$ lines in the plane so that no two are parallel and no three intersect in a common point. What is the number of regions into which these lines partition the plane? Prove.

For example, the lines in the following diagram partition the plane into seven regions:



**Exercise 6** (10 Points) What's wrong with the following proof?

We prove that for any $n \in \mathbb{N}$ and any $a \in \mathbb{R}$, $a^n = 1$. The proof proceeds by strong induction. For the induction basis, $a_0 = 1$ and the claim holds. Assume that the claim holds for all $k$ up to $n$. Then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1$$

This proves the claim.

**Exercise 7** (10 Points) Prove the following about strong induction principle, principle of induction, and well ordering:

(a) Prove the induction principle from the principle of strong induction

(b) Prove the principle of strong from the principle of well ordering

(c) Prove the principle of well ordering from the induction principle

**Exercise 8** (10 Points) Prove the following:

(a) Prove that $\sqrt{3}$ is irrational

(b) Prove that the sum of a rational number and an irrational number results in an irrational number

**Exercise 9** (10 Points) Another general proof technique is *proof by contrapositive.* To prove a statement of the form "If A, then B" it suffices to prove "If not B, then not A". Using the contrapositive technique, prove that if the product of integers $p$ and $q$ is odd, then both $p$ and $q$ must be odd.

**Exercise 10** (10 Points) Given that $a$, $b$, and $c$ are odd integers, prove that $ax^2 + bx + c = 0$ does not have a rational root.