CS103X: Discrete Structures
Problem Bank Pre-midterm

Winter 2008

Exercise 1 (20 points). Let A be a set with m elements and B be a set with n elements, and assume $m < n$. For each of the following sets, give upper and lower bounds on their cardinality and provide sufficient conditions for each bound.

(a) $A \cap B$

(b) $A \cup B$

(c) $A \setminus B$

(d) $2^A \cup A$

Exercise 2 (30 Points). In addition to union ($\cup$), intersection ($\cap$), difference ($\setminus$) and power set (2^A), let us add the following two operations to our dealings with sets:

- Pairwise addition: $A \oplus B := \{a + b : a \in A, b \in B\}$ (This is also called the Minkowski addition of sets A and B.)
- Pairwise multiplication: $A \otimes B := \{a \times b : a \in A, b \in B\}$

For example, if A is $\{1, 2\}$ and B is $\{10, 100\}$, then $A \oplus B = \{11, 12, 101, 102\}$ and $A \otimes B = \{10, 20, 100, 200\}$. Now answer the following questions:

(a) (10 points) Succinctly describe the following sets:

1. $\mathbb{N} \oplus \emptyset$
2. $\mathbb{N} \oplus \mathbb{N}$
3. $\mathbb{N}^+ \oplus \mathbb{N}^+$
4. $\mathbb{N}^+ \otimes \mathbb{N}^+$

(b) (10 points) If E is the set of all positive even numbers, whats the shortest way to write the set of all positive multiples of 4? Of 8?

(c) (10 points) Let $S := \{n^2 : n \in \mathbb{N}^+\}$. A *Pythagorean triple* consists of three positive integers x, y, and z such that $x^2 + y^2 = z^2$. Construct the set of all possible z that could appear as the last element of a Pythagorean triple using only the set S and the set operations we have so far.

Exercise 3 (15 points). Is it possible for every member of a set A to also be a subset of A? If so, is it possible for all cardinalities? Provide positive examples or proofs as to why this cannot be.

Exercise 4 (20 points).

(a) Prove or disprove: $2^A \cap 2^B = 2^{(A \cap B)}$ for any sets A, B.

(b) Prove or disprove: $2^A \cup 2^B = 2^{(A \cup B)}$ for any sets A, B.

Exercise 5. Consider a $2^n \times 2^n$ checkered board (an ordinary chessboard is an 8×8 board) with one square deleted. A triomino is an L-shaped piece composed of 3 squares, i.e. a 2×2 checkered board with one square removed. Show that it is possible to completely cover the rest of the board with non-overlapping triominoes (such a covering is called a tiling).

Exercise 6. Show that every integer multiple of 4 can be expressed as the difference of two perfect squares.

Exercise 7. A perfect number is a natural number $n \geq 2$ with the property that the sum of all of n's divisors (including 1, but not n itself) is n. 6 and 28 are the first two examples. Prove that if $(2p - 1)$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number. Using this property, find another perfect number besides 6 and 28.

Exercise 8. Compute the following without using compuer software. You should find Fermat's Little Theorem useful for some of these.

(a) The last decimal digit of 3^{1000}.

(b) 3^{1000} rem 31.

(c) $3/16$ in $\mathbb{Z}_{31}$

Exercise 9. Show that if a round-robin tournament (every team plays every other team) has an odd number of teams, it is possible for every team to win exactly half its games.

Exercise 10. Prove that the expressions $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y