CS 103X: Discrete Structures
Homework Assignment 7

Due March 3, 2006

Where proofs are required, your writeup should be neat, clear, and concise, while still presenting a complete proof. For combinatorial counting questions, you should state your answer and give a concise derivation, concentrating on the main formulas and conceptual points. Messy and long-winded proofs and solutions will not be accepted.

Exercise 1 (40 points)**.** Prove either algebraically or combinatorially:

(a) For $p, n \geq 0$, $\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1}$

(b) $\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$

(c) $\binom{n}{k}\binom{n-k}{p-k} = \binom{p}{k}\binom{n}{p}$

(d) $\sum_{k=0}^{p} \binom{n}{k}\binom{n-k}{p-k} = 2^p \binom{n}{p}$

Exercise 2 (10 points)**.** Simplify the following to a closed-form expression (without summation):

$$\sum_{k=0}^{n} 2^k \binom{n}{k}.$$

Exercise 3 (20 points)**.** Prove the inclusion-exclusion principle by induction. *Be neat, clear, and concise in your writeup, while still presenting a complete proof.*

Exercise 4 (10 points)**.** Second Silicon Valley question: What is the number of six-figure salaries that are not multiples of either 3, 5, or 7.

Exercise 5 (10 points)**.** What is the number of integer solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 100,$$

such that $0 \leq x_i \leq 40$ for each $1 \leq i \leq 4$.

Exercise 6 (10 points)**.** Austin and Carlos finally hooked up, and are throwing a series of fab wedding bashes, one on each evening of their wedding week. (So seven overall.) They have 60 friends in total (which of course include Anna, Britney, Caitlyn, and Brian), and are inviting 20 of them each evening. How many ways are there to distribute the invitations, such that each friend gets invited at least once? Your answer may consist of a summation.

Exercise 7 (Extra credit:)**.** Give a combinatorial proof of the following identity:

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n.$$